FE-I-FEC101-SK2.

Q.1. A) $\bar{A}^{1} = A^{\theta} = \frac{1}{2}\begin{bmatrix} \sqrt{2} & -i\sqrt{2} & 0 \\ i\sqrt{2} & -\sqrt{2} & 0 \\ 0 & 0 & 2 \end{bmatrix}$ ——— (1)

$AA^{\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$ ——— (3)

——— (4)

A is unitary.

Hence, $\bar{A}^{1} = A^{\theta} = \frac{1}{2}\begin{bmatrix} \sqrt{2} & -i\sqrt{2} & 0 \\ i\sqrt{2} & -\sqrt{2} & 0 \\ 0 & 0 & 2 \end{bmatrix}$ ——— (5)

B) $1+i = \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \sqrt{2}\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right]$ ——— (1)

$(1+i)^{100} = (\sqrt{2})^{100}\left[\cos 25\pi + i\sin 25\pi\right]$ ——— (2)

$= 2^{50}\left[\cos 25\pi + i\sin 25\pi\right]$ ——— (3)

$(1-i)^{100} = 2^{50}\left[\cos 25\pi - i\sin 25\pi\right]$ ——— (4)

$\therefore (1+i)^{100} + (1-i)^{100} = 2^{50}\left(2\cdot\cos 25\pi\right)$

$= 2^{51}(-1)$

$= -2^{51}$ ——— (5)

(C) $y = f(x) = e^{x\cos x}$

$y_1 = f'(x) = e^{x\cos x}(-x\sin x + \cos x)$ ——— (1)

$y_2 = e^{x\cos x}(-x\cos x - \sin x - \sin x)$
$\quad + (-x\sin x + \cos x)\, y_1(x)$ ——— (2)

$y(0) = e^{0\cos 0} = e^0 = 1$ ——— (3)

$y_1(0) = e^0(0+1) = 1$ ——— (4)

$y_2(0) = e^0(0) + (1)\, y_1(0) = 1$ ——— (5)

$e^{x\cos x} = 1 + x + \frac{1}{2}x^2 + \cdots$

(D) $u_x = -\frac{1}{2}u^3 (1 - 2xy + y^2)^{-3/2}(-2y)$

$= \frac{-1}{\cancel{2}} u^3 (-\cancel{2}y) = yu^3$

$\therefore x u_x = xyu^3$

$u_y = -\frac{1}{2} u^3 (-2x + 2y)$

$y u_y = -\frac{1}{\cancel{2}} u^3 (-\cancel{2})(x - y)\, y$

$= xyu^3 - y^2u^3$

$x u_x - y u_y = \cancel{xy u^3} - \cancel{xy u^3} + y^2u^3 = y^2u^3$

E $y = \frac{2}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$

$2 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$ —— (1)

put $x = 1, 2, 3$, we get

$A = 1, \quad B = -2, \quad C = 1$ —— (3)

$y = \frac{1}{x-1} - \frac{2}{x-2} + \frac{1}{x-3}$ —— (4)

$y_n = (-1)^n \cdot n! \left[\frac{1}{(x-1)^{n+1}} - \frac{1}{(x-2)^{n+1}} + \frac{1}{(x-3)^{n+1}} \right]$ —— (5)

F $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}, \quad x_0 = 2$

$x_1 = 2 - \frac{[2^3 - 2(2) - 5]}{3(2)^2 - 2}$ —— (1)

$= 2 - \frac{(-1)}{10} = 2 + \frac{1}{10} = 2.1$ —— (2)

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2.1 - \frac{f(2.1)}{f'(2.1)}$

$x_2 = 2.1 - \frac{0.061}{11.23} = 2.0946$ —— (3)

$x_3 = 2.0946 - \frac{f(2.0946)}{f'(2.0946)}$ —— (4)

$= 2.0946 - \frac{0.0005}{11.162} = 2.0946$

$\therefore x = 2.0946$ —— (5).

Q. 2. $\boxed{A}$

$$\xrightarrow{R_{12}} \begin{bmatrix} 1 & 0 & 4 & 3 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

$$\xrightarrow[R_4 - R_1]{R_3 - 3R_1} \begin{bmatrix} 1 & 0 & 4 & 3 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -12 & -7 \\ 0 & 1 & -6 & -3 \end{bmatrix}$$

$$\xrightarrow[R_4 - R_2]{R_3 - R_2} \begin{bmatrix} 1 & 0 & 4 & 3 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & -9 & -6 \\ 0 & 0 & -3 & -2 \end{bmatrix}$$

$$\xrightarrow[C_4 - 3C_1]{C_3 - 4C_1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & -9 & -6 \\ 0 & 0 & -3 & -2 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{3} R_3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & -3 & -2 \\ 0 & 0 & -3 & -2 \end{bmatrix}$$

$$\xrightarrow[R_4 - R_3]{\cancel{R_4 - R_2}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & -3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow[C_4 + C_2]{C_3 + 3C_2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[-\frac{1}{2} C_2]{-\frac{1}{3} C_3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{C_4 - C_3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\underline{\rho(A) = 3}$$

B) $x^5 = 1+i$

$$= \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) \quad \text{---(1)}$$

$$= \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \quad \text{---(2)}$$

$$x^5 = \sqrt{2}\left[\cos\left(2k\pi + \frac{\pi}{4}\right) + i\sin\left(2k\pi + \frac{\pi}{4}\right)\right] \quad \text{---(3)}$$

$$x^5 = \sqrt{2}\left[\cos(8k+1)\frac{\pi}{4} + i\sin(8k+1)\frac{\pi}{4}\right]$$

$$x = 2^{1/10}\left[\cos(8k+1)\frac{\pi}{4} + i\sin(8k+1)\frac{\pi}{4}\right]^{1/5} \quad \text{---(4)}$$

$$= 2^{1/10}\left[\cos(8k+1)\frac{\pi}{20} + i\sin(8k+1)\frac{\pi}{20}\right]$$

put $k = 0, 1, 2, 3, 4$, we get ---(5)

x_0, x_1, x_2, x_3, x_4 ---(6)

continued product $= x_0 x_1 x_2 x_3 x_4$

$$= 2^{5/10} \cdot e^{i\left(\frac{1+9+17+25+33}{20}\right)\pi} \quad \text{---(7)}$$

$$= 2^{1/2}\, e^{i\left(\frac{85\pi}{20}\right)} = 2^{1/2}\left(e^{i\frac{17\pi}{20}}\right)$$

$$= \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \underline{\underline{1+i}} \quad \text{---(10)}$$

1(c) $u(x,y) = \operatorname{cosec}^{-1}\left(\frac{x^{1/3}+y^{1/2}}{x^{1/3}+y^{1/3}}\right)^{1/2}$

*

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* let $u = \operatorname{cosec} u = f(u) = \frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}$ —— (1)

$u(xt, yt) = t^{1/12}\, u(x,y)$ —— (2)

u is homo. fn with $n = \frac{1}{12}$ —— (3)

$x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} = g(u)[g'(u)-1]$ —— (1)

$g(u) = \frac{n f(u)}{f'(u)} = \frac{1}{12} \cdot \frac{\operatorname{cosec} u}{-\operatorname{cosec} u \cdot \cot u}$ —— (4)

$= -\frac{1}{12} \tan u$ —— (6)

$g'(u) = -\frac{1}{12}(\sec^2 u)$

$= -\frac{1}{12}(\tan^2 u + 1)$

$= -\frac{1}{12}\tan^2 u - \frac{1}{12}$ —— (8)

By corollary

$x^2 u_{xx} + 2xy\, u_{xy} + u_{yy} = -\frac{1}{12}\tan u \left[-\frac{1}{12}\tan^2 u - \frac{1}{12} - 1\right]$

$= -\frac{1}{12}\tan u \left[-\frac{1}{12}\tan^2 u - \frac{13}{12}\right]$

$= \frac{1}{144}\tan u\,(\tan^2 u + 13)$ —— (10)

(D) $y = a\cos(\log x) + b\sin(\log x)$

$$y_1 = \frac{-a\sin(\log x)}{x} + \frac{b}{x}\cos(\log x) \quad \text{—— (1)}$$

$$xy_1 = -a\sin(\log x) + b\cos(\log x) \quad \text{—— (2)}$$

$$xy_2 + y_1(1) = \frac{-a\cos(\log x)}{x} - \frac{b\sin(\log x)}{x} \quad \text{—— (3)}$$

$$x^2y_2 + xy_1 = -(a\cos(\log x) + b\sin\log x)$$

$$x^2y_2 + xy_1 + y = 0 \quad \text{—— (5)}$$

$$x^2y_{n+2} + n(2x)y_{n+1} + \frac{n(n-1)}{\cancel{2!}}\cancel{(2)}\,y_n$$

$$+\,xy_{n+1} + n\cdot 1\cdot y_n + y_n = 0 \quad \text{—— (7)}$$

$$x^2y_{n+2} + (2n+1)xy_{n+1} + (n^2 - \cancel{n} + \cancel{n} + 1)y_n = 0$$

$$x^2y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0 \quad \text{—— (10)}$$

(E)

$$x = \frac{1}{10}(12 - y - z)$$

$$y = \frac{1}{10}(13 - 2x - z) \quad \text{—— 12}$$

$$z = \frac{1}{10}(14 - 2x - 2y)$$

(M)

		x	y	z
4 →	1	1.2	1.06	0.948
6 →	2	0.9992	1.0054	0.9991
8 →	3	0.9996	1.0002	1
10 →	4	1	1	1

(F) i) $\tanh(\log x)$

$$= \frac{e^{\log x} - e^{-\log x}}{e^{\log x} + e^{-\log x}} \quad \text{--- (1)}$$

$$= \frac{x - x^{-1}}{x + x^{-1}} = \frac{x - \frac{1}{x}}{x + \frac{1}{x}} = \frac{x^2 - 1}{x^2 + 1} \quad \text{--- (4)}$$

(2) (3)

Put $x = \sqrt{3}$

$$= \frac{3-1}{3+1} = \frac{2}{4} = \frac{1}{2} \quad \text{--- (5)}$$

ii) Let $\sinh^{-1} x = y$

$$x = \sinh y \quad \text{--- (A)} \quad \text{--- (1)}$$

$$1 + x^2 = 1 + \sinh^2 y = \cosh^2 y \quad \text{--- (2)}$$

$$\sqrt{1+x^2} = \cosh y \quad \text{--- (3)}$$

$$y = \cosh^{-1} \sqrt{1+x^2} \quad \text{--- (B)}$$

By (A) and (B)

$$\sinh^{-1} x = \cosh^{-1} \sqrt{1+x^2} \quad \text{--- (5)}$$

Q3 (A) $AX = B$.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$$

$R_2 - R_1,\ R_3 - R_1$ — (1)

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 3 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$$ — (2)

$R_3 - 3R_2$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$ — (3)

$2z = 0 \Rightarrow z = 0$

$y + 2z = 1 \Rightarrow y = 1$

$x + y + z = 3$ — (5)

$x + 1 + 0 = 3$ $\boxed{x = 2}$

$\boxed{r = n}$, $\rho(A) = \rho(A|B)$

$\Rightarrow$ consistent, unique sol. — (4)

B

$$\log\left(\frac{a-bi}{a+bi}\right) = \log(a-bi) - \log(a+bi)$$ — (1)

$$= \cancel{\tfrac{1}{2}\log(a^2+b^2)} - i\tan^{-1}\frac{b}{a} - \cancel{\tfrac{1}{2}\log(a^2+b^2)} - i\tan^{-1}\frac{b}{a}$$ — (2)

$$= -2i\tan^{-1}\frac{b}{a}$$ — (3)

$\therefore$ L.H.S $= \tan\left\{i\left(-2i\tan^{-1}\frac{b}{a}\right)\right\} = \tan\left\{2\tan^{-1}\frac{b}{a}\right\}$ — (3)

$= \tan 2\alpha \quad , \quad \alpha = \tan^{-1}\frac{b}{a}$

$= \frac{2\tan\alpha}{1+\tan^2\alpha} \longrightarrow (4) \qquad \tan\alpha = b/a$

$$= \frac{2\,b/a}{1+b^2/a^2} = \frac{2\,b/a \times a^2}{a^2+b^2} = \frac{2ab}{a^2+b^2}$$

$\longrightarrow (5)$

1c $f(x) = 2x^3 + 7x^2 + x - 1 \;,\; f(2) = 45$

$f'(x) = 6x^2 + 14x + 1 \quad , \quad f'(2) = 53$

$f''(x) = 12x + 14 \quad , \quad f''(2) = 38$

$f'''(x) = 12 \quad , \quad f'''(2) = 12$

$f(x) = f(2) + (x-2) f'(2) + \frac{(x-2)^2}{2!} f''(2) + \ldots$

$f(x) = 45 + (x-2)\,53 + \frac{(x-2)^2}{2!}(38) + \frac{(x-2)^3}{3!}(12)$

$= 45 + 53(x-2) + 19(x-2)^2 + 2(x-2)^3$

Q.9.D. $u = f(l, m, n)$

let $l = 2x - 3y$

$m = 3y - 4z$

$n = 4z - 2x$

$$\frac{\partial l}{\partial x} = 2, \quad \frac{\partial l}{\partial y} = -3, \quad \frac{\partial l}{\partial z} = 0$$

$$\frac{\partial m}{\partial x} = 0, \quad \frac{\partial m}{\partial y} = 3, \quad \frac{\partial m}{\partial z} = -4$$

$$\frac{\partial n}{\partial x} = -2, \quad \frac{\partial n}{\partial y} = 0, \quad \frac{\partial n}{\partial z} = 4.$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial l}(2) + \frac{\partial u}{\partial m}(0) + \frac{\partial u}{\partial n}(-2)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial l}(-3) + \frac{\partial u}{\partial m}(3) + \frac{\partial u}{\partial n}(0)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial l}(0) + \frac{\partial u}{\partial m}(-4) + \frac{\partial u}{\partial n}(4)$$

$$6\frac{\partial u}{\partial x} + 4\frac{\partial u}{\partial y} + 3\frac{\partial u}{\partial z}$$

$$= \cancel{12\frac{\partial u}{\partial l}} \cancel{-12\frac{\partial u}{\partial m}} \cancel{-12\frac{\partial u}{\partial l}} + \cancel{12\frac{\partial u}{\partial m}}$$

$$\cancel{-12\frac{\partial u}{\partial m}} + \cancel{12\frac{\partial u}{\partial n}} = 0.$$

$\boxed{E}$ $u = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$

$p = u_x = 3x^2 + 3y^2 - 30x + 72$

$r = u_{xx} = 6x - 30$

$q = u_y = \cancel{30}\ 6xy - 30y$

$t = u_{yy} = 6x - 30$, $s = u_{xy} = 6y$ ← (2)

points	r	s	t	$rt - s^2$	conclusion
$(6,0)$	$6 > 0$	0	6	$36 > 0$	u_{min}
$(4,0)$	$-6 < 0$	0	-6	$36 > 0$	u_{max}

← (4)

$\therefore$ ~~u_{min}~~ $u_{max} = \cancel{108}, 112$, at $(4,0)$

$u_{min} = 108$, at $(6,0)$ ← (5)

$\boxed{11}$ Let $y = a + bx + cx^2$ — (1)

$N = 5$

$$\left.\begin{aligned} &\Sigma x = 10,\ \Sigma y = 12.9,\ \Sigma x^2 = 30, \\ &\Sigma x^3 = 100,\ \Sigma x^4 = 354, \\ &\Sigma xy = 37.1,\ \Sigma x^2 y = 130.3 \end{aligned}\right\} \text{— (4)}$$

$12.9 = 5a + 10b + 30c$ — (5)

$37.1 = 10a + 30b + 100c$ (6)

$130.3 = 10a + 100b + 354c$ (7)

$a = 1.42,\ b = -1.07,\ c = 0.55$

$y = 1.42 - 1.07x + 0.55x^2$